

The generalized second law in phantom dominated universes in the presence of black holes

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Abstract

This Letter considers the generalized second law of gravitational thermodynamics in two scenarios featuring a phantom dominated expansion plus a black hole. The law is violated in both scenarios.

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Current observational evidence leaves enough room for the dark energy field driving the present accelerated expansion of the Universe to be of phantom type [1] -a form of energy that violates the dominant energy condition (DEC). If future experiments come to confirm this, it will entail a profound impact on cosmology and field theory. On the one hand, fields violating the DEC (i.e., satisfying $\rho + p < 0$) face quantum instabilities [2] and may drive, under certain conditions, the scale and Hubble factors as well as the curvature to diverge in a finite time ripping apart every bound structure, from galaxy clusters down to atomic nuclei [3]. On the other hand, as shown by Babichev *et al.* [4], black holes by accreting phantom energy lose mass and eventually disappear altogether. This is easy to understand. In a general accreting process the black hole mass rate is proportional to $A(\rho + p)$, where A is the area of the black hole horizon and ρ and p are the energy and pressure of the accreted fluid, respectively. If the latter fulfills the DEC, the black hole mass will increase otherwise it will decrease. In this second case, the immediate consequence is that the area of the black hole horizon will go down along with its entropy, which is given by $S_{BH} = 4\pi M^2$ with M the mass of the (Schwarschild) black hole[15].

A somewhat similar situation arises in the process of Hawking radiation. There the decrease of the black hole mass can be traced to the accretion of virtual particles of negative energy. However, as shown by Bekenstein [5], the overall entropy does not diminish for the emitted radiation offsets for the loss of black hole entropy. Nevertheless, in the case contemplated by Babichev *et al.* there is no particle emission to make up for the decrease of the black hole entropy. Thus, unless it gets compensated by a corresponding increase in the area of the future cosmic horizon and/or the phantom entropy, the generalized second law (GSL) of gravitational thermodynamics will be violated. (The latter, asserts that the entropy of matter and fields inside the horizon plus the entropy of the horizon is a non-decreasing quantity). The target of this Letter is to explore this.

Before going any further, it is expedient to recall that future event horizon possess an entropy proportional to its area. In a strict sense this has been proven rigourously just for the de Sitter horizon [6]. However, it is only natural to associate an entropy to the horizon area as it measures our lack of knowledge about what is going on beyond it. This is why the alluded proportionality is believed to hold true also in more general space-times [7, 8].

Babichev *et al.* consider a spatially flat Friedmann–Robertson–Walker universe filled by a phantom fluid, of energy density and pressure $\rho = -\frac{1}{2}\dot{\phi}^2 + V(\phi)$ and $p = -\frac{1}{2}\dot{\phi}^2 - V(\phi)$,

respectively, that dominates the expansion, and a Schwarzschild black hole. If this is massive enough, its mass decrease via Hawking radiation (proportional to M^{-2}) can be safely ignored and one can write $\dot{M} = -16\pi M^2 \dot{\phi}^2$ regardless of the phantom potential. The latter is felt on \dot{M} through the field rate only.

In a recent paper [9], we demonstrated that in a universe dominated by phantom energy the GSL is satisfied at least in two separate scenarios: (i) When the equation of state parameter of the fluid, $w \equiv p/\rho$, is a constant and, (ii) when the potential follows the power law $V(\phi) = V_0 \phi^\alpha$ with α a constant parameter lying in the range $0 < \alpha \leq 4$ [10]. In both cases, it was found that the entropy of the phantom fluid is negative[16], augments with the expansion of the universe and that the entropy of the future cosmic horizon diminishes. The latter can be written as $S_H = \mathcal{A}/4$, with $\mathcal{A} = 4\pi R_H^2$ the area of the horizon. In general, the radius of the future cosmic horizon, $R_H = a(t) \int_t^\infty dt'/a(t')$, must be calculated via the scale factor of the metric which, in its turn, is governed by the matter and fields filling the universe.

In the first scenario ($w = \text{constant} < -1$, and $V(\phi) \propto \exp(-\lambda\phi)$ with $\lambda = 4\sqrt{\pi/n}$), the corresponding scale factor obeys $a(t) \propto (t_* - t)^{-n}$ where $t \leq t_*$ and $0 < n = -\frac{2}{3(1+w)}$, with t_* being the big rip time [3]. As a result, $R_H = (t_* - t)/n$ whence the entropy of the cosmic horizon diminishes. Using Gibb's equation and assuming thermal equilibrium between the phantom fluid and the horizon it follows that the entropy of the former inside the horizon obeys $S(t) = -S_H(t)$ -see Ref.[9] for details. In consequence, the GSL is preserved but it also follows that if a black hole is present, the GSL will no longer stand, i.e., we will have $\dot{S} + \dot{S}_H + \dot{S}_{BH} < 0$ instead. Obviously, the black hole must be small enough not to significantly alter the scale factor quoted above for it was calculated on the premise that the only energy source of the Einstein field equations was the phantom field.

In the second scenario, the phantom potential follows the power law of above and the radius of the future cosmic horizon is given by

$$R_H = x^{\alpha/4} \Gamma\left(1 - \frac{\alpha}{4}, x\right) \frac{e^x}{H} \quad \left(x \equiv \frac{4\pi}{\alpha} \phi^2\right), \quad (1)$$

where

$$H = \sqrt{\frac{2V_0}{3}} (4\pi)^{\frac{1}{2}-\frac{\alpha}{4}} \alpha^{\alpha/4} x^{\alpha/4} \quad (2)$$

is the Hubble factor, $\dot{a}(t)/a(t)$, and $\Gamma(1 - \frac{\alpha}{4}, x)$ the incomplete gamma function. As a consequence, R_H diminishes as the universe expands. In its turn, the scalar field reads

$$\phi(t) = \left[\phi_i^{\frac{4-\alpha}{2}} + \sqrt{\frac{V_0}{24\pi}} \frac{(4-\alpha)\alpha}{2} (t - t_i) \right]^{\frac{2}{4-\alpha}} \quad (0 < \alpha < 4), \quad (3)$$

$$\phi(t) = \phi_i \exp \left[4 \sqrt{\frac{V_0}{24\pi}} (t - t_i) \right] \quad (\alpha = 4). \quad (4)$$

Note that in both cases $\dot{\phi} > 0$. Here the i subscript indicates that the corresponding quantity is to be evaluated at some suitable initial time, e.g. when the phantom energy begins to dominate the expansion.

The time derivative of the horizon plus the phantom fluid can be written as

$$\dot{S} + \dot{S}_H = \pi \sqrt{\frac{3}{2V_0}} (4\pi)^{-\frac{1}{2}+\frac{\alpha}{4}} \alpha^{-\alpha/4} R_H H \left[\Gamma\left(\frac{4-\alpha}{4}, x\right) e^x \left(2 + \frac{\alpha}{2x}\right) - 2x^{-\alpha/4} \right], \quad (5)$$

which is a positive-definite quantity for any finite x .

From the definition of x it is readily seen that $\dot{\phi} = \sqrt{\alpha/(4\pi)} H/(2\sqrt{x})$, and in virtue of this equation the black hole mass rate is

$$\dot{M} = -\alpha \frac{H^2}{x} M^2, \quad (6)$$

thereby

$$\dot{S}_{BH} = -8\pi\alpha \frac{H^2}{x} M^3. \quad (7)$$

The GSL, $\dot{S}_{BH} + \dot{S} + \dot{S}_H \geq 0$, will be satisfied provided the black hole mass does not exceed the critical value

$$M_{cr} = \sqrt{\frac{3}{8V_0}} (4\pi)^{-\frac{1}{2} + \frac{\alpha}{4}} \alpha^{-\frac{1}{3} - \frac{\alpha}{4}} x^{\frac{1}{3} - \frac{\alpha}{12}} \times \left\{ e^x \Gamma\left(\frac{4-\alpha}{4}, x\right) \left[\Gamma\left(\frac{4-\alpha}{4}, x\right) e^x \left(2 + \frac{\alpha}{2x}\right) - 2x^{-\alpha/4} \right] \right\}^{\frac{1}{3}}. \quad (8)$$

As inspection reveals, M_{cr} decreases with ϕ^2 (and therefore with time) at fixed α . This stems from the fact that $\dot{S} + \dot{S}_H$ is a decreasing function.

Again, the the black hole mass has to be low enough to not significantly modify the scale factor nor the cosmic horizon radius, i.e., we must have $M \ll E_\phi$, where $E_\phi = \frac{4\pi}{3} R_H^3 \rho$ is the energy of the phantom fluid inside the horizon. In terms of x it reads,

$$E_\phi = \sqrt{\frac{3}{8V_0}} (4\pi)^{-\frac{1}{2} + \frac{\alpha}{4}} \alpha^{-\frac{\alpha}{4}} x^{\frac{\alpha}{2}} \left[e^x \Gamma\left(\frac{4-\alpha}{4}, x\right) \right]^3. \quad (9)$$

To most direct way to see whether the GSL is violated or not is to compare the evolution of E_ϕ , M_{cr} , and M . The expression for the latter follows from integrating Eq. (6):

$$M = \frac{M_i}{1 + M_i \sqrt{\frac{2V_0}{3}} (4\pi)^{\frac{1}{2} - \frac{\alpha}{4}} \alpha^{\alpha/4} (x^{\alpha/4} - x_i^{\alpha/4})}, \quad (10)$$

with $M_i \equiv M(x = x_i)$.

Inspection of the last tree equations alongside their expressions for $x \gg 1$, namely, $E_\phi \sim x^{-\alpha/4} + \mathcal{O}(x^{-\frac{\alpha}{4}-3})$, and $M_{cr} \sim x^{-\frac{1}{3} - \frac{\alpha}{4}} + \mathcal{O}(x^{-\frac{\alpha}{4} - \frac{2}{3}})$, which follow from the relation [12]

$$e^x \Gamma\left(1 - \frac{\alpha}{4}, x\right) \sim x^{-\alpha/4} \left(1 - \frac{\alpha}{4x} + \frac{\alpha^2 + 4\alpha}{16x^2} + \dots\right),$$

reveals that for reasonable values of M_i , initially (i.e., for $x_i \sim \mathcal{O}(1)$) we will simultaneously have $M < M_{cr}$ and $M \ll E_\phi$, and somewhat later we will have $M_{cr} < M \ll E_\phi$, instead. The latter corresponds to a violation of the GSL.

Indeed, as Fig. 1 shows, for every α there is an initial x interval in which M_{cr} is larger than M . The GSL is fulfilled in that interval. However, later on, M_{cr} becomes smaller than M (with $M \ll E_\phi$) and the GSL is violated. Still further ahead, M is no longer negligible against E_ϕ . Thus, for sufficiently large x , this assumption breaks down and from there on it cannot be said whether the GSL is violated or not. At any rate, for any α , there is an ample x interval in which it can be safely said that the GSL does not hold.

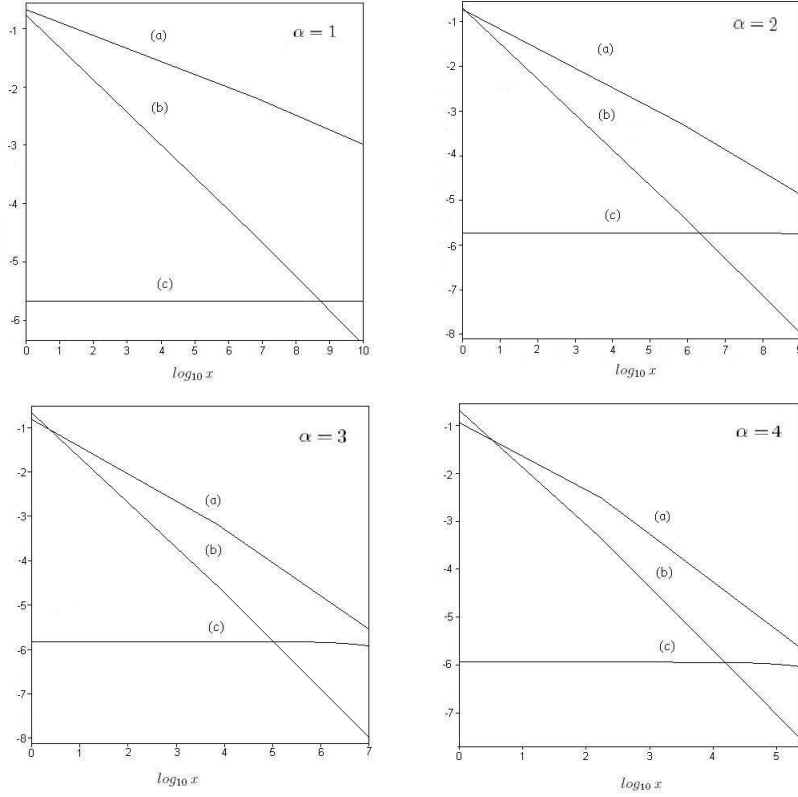


FIG. 1: Evolution of E_ϕ (a), M_{cr} (b), and M (c), *vs* x in logarithmic scales. The initial mass of the black hole was chosen $M_i = 10^{-5}E_\phi(x_i)$. As it can be appreciated in the four panels, the black hole mass exhibits a much steeper decrease than the critical mass. Consequently, irrespective of the M_i value, sooner or later we will have that $M \geq M_{cr}$ and thus the GSL will fail. Later on, M will become non-negligible (compared to E_ϕ) and our model will no be longer valid. In plotting the graphs we have set $V_0^{-1/2} = 1$, and $x_i = 1$, for simplicity.

Interestingly enough, while in the second scenario, which is big rip free, there is enough room for the GSL to be fulfilled there is no room whatsoever in the first scenario, which features a big rip.

In the light of the foregoing results some reactions may arise: (a) Some phantom energy fields might be physical but not those considered in this Letter. In fact, some predictions lending support to phantom fields may have come from an erroneous interpretation of the observational data [13]. (b) The GSL was initially formulated for systems fulfilling the DEC, so there is no reason why it ought to be satisfied for systems that violate it. What is more,

in a strict sense, a general proof of the GSL even for systems complying with the DEC is still lacking [14], therefore there should be no wonder that it fails in some specific instances.

It is for the reader to decide which of these views, if any, is more to his/her liking.

Yet, one may argue that it is unclear that black holes retain their thermodynamic properties (entropy and temperature) in presence of a field that does not comply with the DEC. In such an instance, one may think, that there is no room for the black hole entropy in the expression for the GSL. However, the latter is often formulated by replacing S_{BH} by the black hole area. Again, this variant of the GSL will fail in the two cases of above. In all, if eventually phantom energy is proven to be a physical reality, it will mean a serious threat to the generalized second law.

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- [15] We use units in which $c = G = \hbar = k_B = 1$.
- [16] The negative character of phantom's entropy was previously noted in Refs. [11].